J. DIFFERENTIAL GEOMETRY
39 (1994) 139-172

SUBVARIETIES OF GENERAL HYPERSURFACES IN PROJECTIVE SPACE

GENG XU

0. Introduction

We are interested in the following question: If C is an irreducible curve (possibly singular) on a generic surface of degree d in a projective 3-space $\mathbf{P}^3$, can the geometric genus of C (the genus of the desingularization of C) be bound from below in terms of d? Bogomolov and Mumford [14] have proved that there is a rational curve and a family of elliptic curves on every K-3 surface. Since a smooth quartic surface in $\mathbf{P}^3$ is a K-3 surface, there are rational and elliptic curves on a generic quartic surface in $\mathbf{P}^3$. On the other hand, Harris conjectured that on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ there are neither rational nor elliptic curves.

Now let C be a curve on a surface S of degree d in $\mathbf{P}^3$. By the Noether-Lefschetz Theorem, if $d \geq 4$ and S is generic, then C must be a complete intersection of S with another surface S_1 of degree k. In this case we say that C is a type (d, k) curve on S. Clemens [4] has proved that there is no type (d, k) curve with geometric genus $g \leq \frac{1}{2}dk(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$; in particular, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$.

Our first main result is the following.

Theorem 1. *On a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-3) - 3$, and this bound is sharp. Moreover this sharp bound can be achieved only by a tritangent hyperplane section if $d \geq 6$.*

We immediately conclude that the above conjecture of Harris is true. Meanwhile it is not hard to see that for a generic surface S of degree d in $\mathbf{P}^3$, there is a tritangent hyperplane H and thus $C = H \cap S$ has three double points. Since $\pi(C) = \frac{1}{2}(C \cdot C + K_S \cdot C) + 1 = \frac{1}{2}d(d-3) + 1$, and an ordinary double point drops the genus of a curve by 1, the above bound is sharp.

Received January 11, 1993. Partially supported by Sloan Doctoral Dissertation Fellowship.

Let C be a curve on a generic surface S of degree d in $\mathbf{P}^3$. The main point of the proof of Theorem 1 is to see how bad the singularities of such a curve C can be. We first study the deformation of C at the singular points of C, and obtain that if there is a type (d, k) curve C with certain geometric genus g on a generic surface S of degree d, then there are some homogeneous polynomials vanishing at the singular points of C to a certain expected order. By a Koszul type of argument, we can reduce the degree of these homogeneous polynomials. From these we get control over the singularities of C and obtain Theorem 2.1 which is just a slight improvement of Clemens' results (cf. [3], [4]). Then to prove Theorem 1 in the case $d \geq 6$, it remains only to see what kind of singularities a hyperplane section of S can afford.

We can generalize the above result in $\mathbf{P}^3$ to higher dimensions.

Theorem 2. *Let V be a generic hypersurface of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$, $M \subset V$ a reduced and irreducible divisor, and $p_g(M)$ the geometric genus of the desingularization of M. Then*

$$(0.1)\quad p_g(M) \geq \min\left\{\binom{d-2}{n+1} - \binom{d-4}{n+1} + 1,\ \binom{d}{n+1} - \binom{d-1}{n+1}\right\}.$$

Moreover if

$$(0.2)\qquad \binom{d-2}{n+1} - \binom{d-4}{n+1} + 1 \geq \binom{d}{n+1} - \binom{d-1}{n+1},$$

then the bound

$$(0.3)\qquad p_g(\mathbf{M}) \geq \binom{d}{n+1} - \binom{d-1}{n+1}$$

is sharp, and this sharp bound can be achieved only by a hyperplane section for the case where the inequality holds in (0.2).

Remark. The inequality (0.2) is true when $d \geq C(n)$. For example, $C(3) = 14$, $C(4) = 19$.

If $M \subset V$ as in Theorem 2, then it is well known that M is a complete intersection of V with another hypersurface of degree k. Ein (cf. [5], [6]) has proved that

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-2-k}{n+1}$$

in this case, and his results have generalized to varieties of higher codimensions. Therefore the improvement we make here is in the case $k = 1$.

When $n = 3$ Theorem 2 implies that $p_g(M) \geq 2$ if $d \geq 6$. In case $d = 5$, there is a very interesting conjecture.

Clemens' Conjecture. On a generic quintic 3-fold in a projective 4-space $\mathbf{P}^4$, there are only finite number of rational curves in each degree.

This assertion has been proved by Katz for degree up to 7 (cf. [7], [13], [15]). Mark Green has asked the following:

Question. Does every surface on a generic quintic 3-fold in $\mathbf{P}^4$ have positive geometric genus?

If V is a generic quintic 3-fold, since any one-parameter family of rational curves on V sweeps out a surface of geometric genus 0, an affirmative answer to Green's question will imply Clemens' conjecture.

This paper is organized as follows. We introduce a certain type of singularity in §1. In §2 we state and prove Theorem 2.1, which will be used in the next section. In §3 we prove Theorem 1. Section 4 is devoted to the proof of Theorem 2. In the last section we outline a proof of Proposition 4 which states that a hyperplane section of a generic hypersurface can only have very mild singularities.

Throughout this paper we work over the complex number field $\mathbb{C}$.

I am grateful to my thesis advisor Mark Green for his advice and encouragement, and to David Gieseker, János Kollár, Shigefumi Mori and Jonathan Wahl for valuable discussions about singularities. I am also indebted to Herbert Clemens, Lawrence Ein, and Robert Lazarsfeld for their generous help and illuminating conversations, and to Lawrence Green for his careful reading of the whole paper.

1. Weak type δ singularities

In this section, we introduce a type of singularity, establish some of its elementary properties,and show its relationship with the canonical divisor.

Let V be an n-dimensional smooth variety, and $M \subset V$ be an irreducible codimension-1 singular subvariety. According to Hironaka [11], there is a desingularization of M: $V_{m+1} \xrightarrow{\pi_{m+1}} V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = V$, so that the proper transform $\widetilde{M}$ of M in V_{m+1} is smooth. Here $V_j \xrightarrow{\pi_j} V_{j-1}$ is the blow-up of V_{j-1} along a ν_{j-1}-dimensional submanifold X_{j-1} with $E_{j-1} \subset V_j$ the exceptional divisor. If X_{j-1} is a μ_{j-1}-fold singular submanifold of the proper transform of M in V_{j-1}, we say that M has a *type* $\mu = (\mu_j, \mathbf{X}_j, \mathbf{E}_j | \mathbf{j} \in \{0, 1, \cdots, \mathbf{m}\})$ *singularity.*

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j | j \in \Gamma)$ singularity, and $\Omega \subset V$ is an open set, then we localize our definition by saying that M has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \Gamma_\Omega = \{j | \exists q \in E_j, \ q$ is an infinitely near point of some $p \in \Omega\})$ singularity on Ω.

Given any resolution of the singularity of $M \subset V$ as above, if $Z \subset V$ is a codimension-1 subvariety, such that

$$\pi_j^*(\cdots(\pi_2^*(\pi_1^*(Z) - \delta_0 E_0) - \delta_1 E_1) - \cdots) - \delta_{j-1} E_{j-1}$$

is an effective divisor for $j = 1, 2, \cdots, m+1$, then we say that Z has a *weak type* $\delta = (\delta_j, \mathbf{X}_j, \mathbf{E}_j | \mathbf{j} \in \{0, 1, \cdots, m\})$ *singularity*. It is easy to see that a type μ singularity implies a weak type μ singularity.

In terms of local coordinates, we assume that M has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity on Ω, and $\{z_1, \cdots, z_n\}$ are coordinates on Ω with X_0 defined by $z_{s+1} = \cdots = z_n = 0$. Let

$$z_1' = z_1, \cdots, z_s = z_s, \quad z_{s+1}' = \frac{z_{s+1}}{z_n}, \cdots, z_{n-1}' = \frac{z_{n-1}}{z_n}, \quad z_n' = z_n$$

be coordinates on the blow-up of Ω along X_0, and $h(z_1, \cdots, z_n)$ be a holomorphic function defined on Ω. Setting

$$\begin{aligned} h(z_1, \cdots, z_n) &= h(z_1', \cdots, z_s', z_{s+1}' z_n', \cdots, z_{n-1}' z_n', z_n') \\ &= (z_n')^\rho h^\sharp(z_1', \cdots, z_n'), \end{aligned}$$

then we say that the variety $\{h(z_1, \cdots, z_n) = 0\}$ on Ω has a weak type $\delta_\Omega = (\delta_j, X_j, E_j | j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity, if $\rho \geq \delta_0$, $h^\sharp$ is holomorphic, and $\{(z_n')^{\rho - \delta_0} h^\sharp(z_1', \cdots, z_n') = 0\}$ has a weak type $(\delta_j, X_j, E_j | j \in \{1, \cdots, m\})$ singularity on the blow-up of Ω along X_0.

The property of having a weak type δ singularity is additive in the following sense: if two varieties $\{h_1(z_1, \cdots, z_n) = 0\}$ and $\{h_2(z_1, \cdots, z_2) = 0\}$ have weak type $\delta_\Omega = (\delta_j, X_j, E_j | j \in \Gamma_\Omega)$ singularities on Ω, then so does the variety $\{h_1 + h_2 = 0\}$. This holds because

$$\begin{aligned} h_1(z_1, \cdots, z_n) &= (z_n')^{l_1} h_1^\sharp(z_1', \cdots, z_n'), \\ h_2(z_1, \cdots, z_n) &= (z_n')^{l_2} h_2^\sharp(z_1', \cdots, z_n') \end{aligned}$$

with $l_1, l_2 \geq \delta_0$, so $\min(l_1, l_2) \geq \delta_0$, and

$$\begin{aligned} &(h_1 + h_2)(z_1, \cdots, z_n) \\ &\quad = (z_n')^{\min(l_1, l_2)}((z_n')^{l_1 - \min(l_1, l_2)} h_1^\sharp(z_1', \cdots z_n') \\ &\qquad + (z_n')^{l_2 - \min(l_1, l_2)} h_2^\sharp(z_1', \cdots, z_n')) \\ &\quad = (z_n')^{\delta_0}((z_n')^{l_1 - \delta_0} h_1^\sharp(z_1', \cdots, z_n') + (z_n')^{l_2 - \delta_0} h_2^\sharp(z_1', \cdots, z_n')). \end{aligned}$$

Since both $\{(z_n')^{l_1-\delta_0}h_1^\sharp(z_1', \cdots, z_n') = 0\}$ and $\{(z_n')^{l_2-\delta_0}h_2^\sharp(z_1', \cdots, z_n') = 0\}$ have weak type $(\delta_j, X_j, E_j|j \in \{1, \cdots, m\})$ singularities on the blow-up of Ω along X_0, by induction

$$\{(z_n')^{l_1-\delta_0}h_1^\sharp(z_1', \cdots, z_n') + (z_n')^{l_2-\delta_0}h_2^\sharp(z_1', \cdots, z_n') = 0\}$$

also has a weak type $(\delta_j, X_j, E_j|j \in \{1, \cdots, m\})$ singularity. Then $\{h_1(z_1, \cdots, z_n) + h_2(z_1, \cdots, z_n) = 0\}$ has a weak type $\delta_\Omega = (\delta_j, X_j, E_j)|j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity on Ω.

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j|j \in \{0, 1, \cdots, m\})$ singularity, and $\widetilde{M}_j$ is the proper transform of M in V_j, then by the adjunction formula,

$$\begin{aligned}
K_{\widetilde{m}} &= K_{\widetilde{M}_{m+1}} \\
&= K_{V_{m+1}} + \widetilde{M}_{m+1} \\
&= \pi_{m+1}^*(K_{V_m}) + (n - \nu_m - 1)E_m + \pi_{m+1}^*(\widetilde{M}_m) - \mu_m E_m \\
&= \pi_{m+1}^*(K_{V_m} + \widetilde{M}_m) - (\mu_m - (n - \nu_m - 1))E_m \qquad (1.1)\\
&= \cdots \\
&= \pi_{m+1}^*(\cdots(\pi_2^*(\pi_1^*(K_V + M) - (\mu_0 - (n - \nu_0 - 1))E_0) \\
&\qquad\qquad - (\mu_1 - (n - \nu_1 - 1))E_1 \cdots) \\
&\quad - (\mu_m - (n - \nu_m - 1))E_m.
\end{aligned}$$

Since $n - \nu_j - 1 \geq 1$, we get

Proposition 1.1. *A section of $K_V \otimes M$ with a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j|j \in \{0, 1, \cdots, m\})$ singularity induces a section of $K_{\widetilde{M}}$.*

Definition. Let $T \subset \mathbb{C}^N$ be an open neighborhood of the origin $0 \in T$. Assuming that $\sigma: M \to T$ is a family of reduced equidimensional algebraic varieties, $M_t = \sigma^{-1}(t)$, then we say that the family M_t is μ-equisingular at $t = 0$ in the sense that we can resolve the singularity of M_t simultaneously, that is, there is a proper morphism $\pi: \widetilde{M} \to M$, so that $\sigma \circ \pi: \widetilde{M} \to T$ is a flat map and $\sigma \circ \pi: \widetilde{M}_t = (\sigma \circ \pi)^{-1}(t) \to M_t$ is a resolution of the singularities of M_t. Moreover, if M_t has a type $\mu(t) = (\mu_j(t), X_j(t), E_j(t)|j \in \Gamma(t))$ singularity with the above resolution, then $\mu_j(t) = \mu_j$ and $\Gamma(t) = \Gamma$ are independent of t, and the exceptional divisors and the singular loci of the desingularization $\widetilde{M}_t \to M_t$ have the same configuration for all t (cf. [16], [17], [18]).

2. Curves on generic surfaces in $\mathbf{P}^3$

Our starting point is the following (cf. [2], [8], [9]).

Noether-Lefschetz Theorem. *Every curve on a generic surface of degree $d \geq 4$ in $\mathbf{P}^3$ is a complete intersection.*

Let C be an irreducible curve on a generic surface $S = \{F = 0\}$ of degree $d \geq 5$ in $\mathbf{P}^3$. Then C is a complete intersection of S with another surface $S_1 = \{G = 0\}$ of degree k, i.e., C is a type (d, k) curve on S. Here we always assume that the generic surface S is smooth, and both $\{F = 0\}$ and $\{F = 0\} \cap \{G = 0\}$ are reduced. First of all, we have the following lower bound estimate on the geometric genus $g(C)$ of C.

Theorem 2.1. *If C is a curve on a generic surface S of degree $d \geq 5$ in $\mathbf{P}^3$, and C is a complete intersection of S with another surface of degree k, then $g(C) \geq \frac{1}{2}dk(d-5)+2$.*

Before we go into the proof of Theorem 2.1, let us first set down our notation.

For P a singular point of $C \subset S$, we use $\mathbf{e}(\mathbf{P}, \mathbf{C})$ to denote the multiplicity of C at P (cf. [12, Chap. 9]), that is, if $\pi \colon W \to S$ is the blow-up of S at P, and E is the exceptional divisor, then $\pi^*C = C^* + e(P, C)E$. Here C^* is the proper transform of C by π. If $\{q_1, \cdots, q_s\} = C^* \cap E$, then the points q_i are said to be the *infinitely near points of* $\mathbf{P}$ on $\mathbf{C}$ *of the first order.* Inductively, infinitely near points of q_i $(i = 1, 2, \cdots, s)$ on C^* of the jth order are said to be the *infinitely near points of* $\mathbf{P}$ *on* $\mathbf{C}$ *of the* $(j+1)$*th order.* We define $e(q_i, C) = e(q_i, C^*)$, and so on.

If P_{0j} $(j = 0, 1, \cdots, n_0)$ are all the singular points on C, P_{ij} $(j = 0, 1, \cdots, n_i)$ are all the infinitely near points on C of the ith order $\mu_{ij} = e(P_{ij}, C)$, and E_{ij} is the exceptional divisor resulting from the blowing up at P_{ij}, then C has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij} | (i, j) \in \Gamma)$ singularity with $\Gamma = \{(i, j) | \mu_{ij} > 1\}$, and

$$g(C) = \pi(C) - \sum_{i,j} \frac{1}{2}\mu_{ij}(\mu_{ij} - 1)$$

$$\frac{1}{2}dk(d+k-4) + 1 - \sum_{i,j} \frac{1}{2}\mu_{ij}(\mu_{ij} - 1).$$

Therefore the key to the proof of Theorem 2.1 is to see how bad the singularities of C may be.

Lemma 2.2. *If $F(z_1, z_2)$ is an analytic function on an open set $\Omega \subset \mathbb{C}^2$ defining a curve C, $P_{00} \in \Omega$ is the only singular point of C, and C has a type $\mu_\Omega = (\mu_{ij}, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}, then the curves*

$\{\partial F/\partial z_1 = 0\}$ and $\{\partial F/\partial z_2 = 0\}$ in Ω have weak type $\mu_\Omega - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularities at P_{00}.

Proof. First of all, we note that the conclusion of Lemma 2.2 is independent of the choice of the local coordinates on Ω. Without loss of generality, we may assume $P_{00} = (0, 0) \in \Omega$, and

$$\xi = z_1, \qquad \eta = z_2/z_1$$

are the new coordinates after blowing up at P_{00}; therefore

$$F(z_1, z_2) = z_1^{\mu_{00}} F^*(\xi, \eta).$$

Here $F^* = 0$ is the equation of the proper transform of the curve $\{F = 0\}$ after blowing up at P_{00}. Now

$$\frac{\partial F}{\partial z_1} = z_1^{\mu_{00}-1} \left(\mu_{00} F^* + \xi \frac{\partial F^*}{\partial \xi} - \eta \frac{\partial F^*}{\partial \eta} \right).$$

Since $\{F^* = 0\}$ has a singularity with fewer steps to resolve at P_{1j}, then by induction, both $\{\partial F^*/\partial \xi = 0\}$ and $\{\partial F^*/\partial \eta = 0\}$ have weak type $(\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega - (0, 0))$ singularities. Therefore by additivity $\{\partial F/\partial z_1 = 0\}$ has a weak type $\mu_\Omega - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}. On the other hand,

$$\frac{\partial F}{\partial z_2} = z_1^{\mu_{00}-1} \frac{\partial F^*}{\partial \eta}.$$

Again we see that $\{\partial F/\partial z_2 = 0\}$ has a weak type $\mu_\Omega - 1 = \mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}. q.e.d.

Lemma 2.2 is a special case of the following.

Lemma 2.3. *If $C_t = \{F_t(z_1, z_2) = 0\}$ is an analytic μ-equisingular family of curves in an open set $\Omega \subset \mathbb{C}^2$, C_t has only one singular point $P_{00}(t)$ in Ω, and C_t has a type $\mu(t)_\Omega = (\mu_{ij}, P_{ij}(t), E_{ij}(t) | (i, j) \in \Gamma_\Omega)$ singularity, then the curve $\{dF_t/dt|_{t=0} = 0\}$ in Ω has a weak type $\mu_\Omega - 1 = (\mu_{ij}(0) - 1, P_{ij}(0), E_{ij}(0) | (i, j) \in \Gamma_\Omega)$ singularity at $P_{00}(0)$.*

Proof. Let $P(t) = (c_1(t), c_2(t))$, and

$$F_t(z_1, z_2) = \sum_{i+j \geq \mu_{00}} a_{ij}(t)(z_1 - c_1(t))^i (z_2 - c_2(t))^j.$$

Then

$$\left.\frac{dF_t}{dt}\right|_{t=0} = - \left. \left\{ \frac{dc_1(t)}{dt} \frac{\partial F_0}{\partial z_1} + \frac{dc_2(t)}{dt} \frac{\partial F_0}{\partial z_2} \right\} \right|_{t=0} + \frac{d}{dt} \left. \left\{ \sum_{i+j \geq \mu_{00}} a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j \right\} \right|_{t=0}.$$

By Lemma 2.2, both $\{\partial F_0/\partial z_1 = 0\}$ and $\{\partial F_0/\partial z_2 = 0\}$ have weak type $\mu_\Omega - 1$ singularities at $P_{00}(0)$.

If we move the singular point $P_{00}(t)$ of $F_t = 0$ to $P_{00}(0)$, we get

$$F_t^* = \sum_{i+j\geq\mu_{00}} a_{ij}(t)(z_1 - c_1(0))^i(z_2 - c_2(0))^j .$$

Now we can blow up simultaneously at $P_{00}(0)$. If we let

$$\xi = z_1 - c_1(0), \qquad \eta = (z_2 - c_2(0))/(z_1 - c_1(0))$$

be the new local coordinates after blowing up, then

$$F_t^* = (z_1 - c_1(0))^{\mu_{00}} F_t^\sharp(\xi, \eta),$$
$$\left.\frac{dF_t^*}{dt}\right|_{t=0} = (z_1 - c_1(0))^{\mu_{00}} \left.\frac{dF_t^\sharp(\xi, \eta)}{dt}\right|_{t=0} .$$

Here $F_t^\sharp$ is still a μ-equisingular family, but has improved singularities. By induction, $\{dF_t^\sharp(\xi, \eta)/dt|_{t=0} = 0\}$ has a weak type $(\mu_{ij}(0) - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_\Omega - (0, 0))$ singularity. By additivity we conclude that $\{dF_t/dt|_{t=0} = 0\}$ has a weak type $\mu_\Omega - 1$ singularity at $P_{00}(0)$.

Lemma 2.4. *Let* $F_t \in H^0(\mathbf{P}^3, \mathcal{O}(d))$, $G_t \in H^0(\mathbf{P}^3, \mathcal{O}(k))$, *and* $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ *be a μ-equisingular family of curves with a type* $\mu(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t)|(i, j) \in \Gamma)$ *singularity. Set* $dF_t/dt|_{t=0} = F'$, *and* $dG_t/dt|_{t=0} = G'$. *If all the surfaces* $F_t = 0$ *are smooth, and* $\partial F_0(P)/\partial Z_i \neq 0$, $Z_i(P) \neq 0$ $(i = 0, 1, 2, 3)$ *at every singular point P of* $C = \{F_0 = 0\} \cap \{G_0 = 0\} = \{F = 0\} \cap \{G = 0\}$, *where* $\{Z_0, Z_1, Z_2, Z_3\}$ *are homogeneous coordinates, then the curve* $\{(\partial F/\partial Z_i)G' - (\partial G/\partial Z_i)F' = 0\}$ *on* $S = \{F = 0\}$ *has a weak type* $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma)$ *singularity.*

Proof. We fix $P = P_{0s}(0)$ for some s, and assume that C_t has a type $\mu_s(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t)|(i, j) \in \Gamma_s)$ singularity at $P(t) = P_{0s}(t)$. Denoting $\{z_1, z_2, z_3\} = \{Z_1/Z_0, Z_2/Z_0, Z_3/Z_0\}$, if we solve the equation $F_t(1, z_1, z_2, z_3) = 0$ near the point $P(t)$, and get $z_3 = \varphi_t(z_1, z_2)$, then we can view C_t as a μ-equisingular family of curves locally defined by the equation $G_t(1, z_1, z_2, \varphi_t(z_1, z_2)) = 0$ in an open set $\Omega \subset \mathbb{C}^2$. By Lemma 2.3, the curve locally defined by the equation

$$\frac{dG_t}{dt}(1, z_1, z_2, \varphi_t(z_1, z_2))|_{t=0} = 0$$

on the surface $S = \{F = 0\}$ has a weak type $\mu_s(0) - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_s)$ singularity at $P(0) = P_{0s}(0)$.

From the equation $F_t(1, z_1, z_2, \varphi_t(z_1, z_2)) = 0$, we get

$$F'(1, z_1, z_2, \varphi_0(z_1, z_2)) + \frac{\partial F}{\partial Z_3}(1, z_1, z_2, \varphi_0(z_1, z_2))\frac{d\varphi_t}{dt}(z_1, z_2)|_{t=0} = 0,$$

and thus

$$\frac{d\varphi_t}{dt}|_{t=0} = -\left(\frac{\partial F}{\partial Z_3}\right)^{-1} F'.$$

We also have

$$\begin{aligned}\frac{dG_t}{dt}(1, z_1, z_2, \varphi_t(z_1, z_2))|_{t=0} &= G' + \frac{\partial G}{\partial Z_3}\frac{d\varphi_t}{dt}|_{t=0} \\ &= G' - \left(\frac{\partial F}{\partial Z_3}\right)^{-1}\left(\frac{\partial G}{\partial Z_3}\right)F'.\end{aligned}$$

Thus the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on the surface S has a weak type $\mu_s(0) - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_s)$ singularity at $P(0) = P_{0s}(0)$. Since s is arbitrary, we conclude that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on surface $S = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma)$ singularity.

Lemma 2.5. *Assume $C = \{F = 0\} \cap \{G = 0\}$ is a curve on a smooth surface $S = \{F = 0\}$ in $\mathbf{P}^3$, $\deg F = d$, $\deg G = k$, and C has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij}|(i, j) \in \Gamma)$ singularity. If $Q \in H^0(\mathbf{P}^3, \mathcal{O}(m))$ is not in the homogeneous polynomial ideal (F, G) generated by F and G, and the curve $\{Q = 0\}$ on S has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij}|(i, j) \in \Gamma)$ singularity, then*

$$\sum_{(i,j)\in\Gamma} \mu_{ij}(\mu_{ij} - 1) \leq dkm.$$

Proof. By Bezout's Theorem, the intersection number $I(Q, G)_F$ of the divisors $\{Q = 0\}$ and $\{G = 0\}$ on $S = \{F = 0\}$ is equal to dkm. Let $P_{0s} = P_{os}(0)$ $(s = 0, 1, \cdots, n_0)$ be all the singular points of C on S, $S_{0,1} \xrightarrow{\pi_{0,1}} S_{0,0} = S$ be the blow-up of S at $P_{0,0}$ with $\widetilde{C}_{0,1}$ the proper transform of $C = \{G = 0\} \cap S$ in $S_{0,1}$ and inductively $S_{0,s+1} \xrightarrow{\pi_{0,s+1}} S_{0,s}$ be the blow-up of $S_{0,s}$ at $P_{0,s}$ with $\widetilde{C}_{0,s+1}$ the proper transform of $\widetilde{C}_{0,s}$ in $S_{0,s+1}$. Then $\pi^*_{0,1}C = \mu_{00}E_{00} + \widetilde{C}_{0,1}$. Since $Q = \{Q = 0\}$ has a weak type $\mu - 1$ singularity, $\pi^*_{0,1}Q - (\mu_{00} - 1)E_{00}$ is an effective divisor in $S_{0,1}$, so

$$\begin{aligned}\widetilde{C}_{0,1}(\pi^*_{0,1}Q-(\mu_{00}-1)E_{00}) &= (\pi^*_{0,1}C-\mu_{00}E_{00})(\pi^*_{0,1}Q-(\mu_{j00}-1)E_{00}) \\ &= C\cdot Q-\mu_{00}(\mu_{00}-1).\end{aligned}$$

Therefore

$$\begin{aligned}I(Q\,,\,G)_F &= C\cdot Q \\ &= \widetilde{C}_{0,1}\cdot(\pi^*_{0,1}Q-(\mu_{00}-1)E_{00})+\mu_{00}(\mu_{00}-1) \\ &= \cdots \\ &= \widetilde{C}_{0,n_0+1}\cdot(\pi^*_{0,n_0+1}(\cdots\pi^*_{0,2}(\pi^*_{0,1}Q-(\mu_{00}-1)E_{00}) \\ &\qquad -(\mu_{01}-1)E_{01})-\cdots-(\mu_{0n_0}-1)E_{0n_0}) \\ &\quad +\sum_{s=0}^{n_0}\mu_{0s}(\mu_{0s}-1).\end{aligned}$$

If we continue the above process on all the infinitely near points on C of the first order, and so on, finally we will get

$$I(Q\,,\,G)_F \geq \sum_{(i,j)\in\Gamma}\mu_{ij}(\mu_{ij}-1).\quad \text{q.e.d.}$$

After these four lemmas, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first fix an integer $d\geq 5$. Let g be the minimum integer so that on a generic surface of degree d in $\mathbf{P}^3$ there is a curve C with geometric genus $g(C)\leq g$. Setting

$$H_{m,g}=\{F\in \mathbf{P}H^0(\mathbf{P}^3\,,\,\mathscr{O}(d))|\text{ there is a degree } m \text{ curve } C\subset\{F=0\} \text{ with } g(C)\leq g\},$$

it is well known that $H_{m,g}\subset\mathbf{P}H^0(\mathbf{P}^3\,,\,\mathscr{O}(d))$ is an algebraic subvariety. By our assumption on g and the Noether-Lefschetz Theorem, the natural map

$$\bigcup_{k=1}^{\infty}H_{dk,g}\to\mathbf{P}H^0(\mathbf{P}^3\,,\,\mathscr{O}(d))$$

is surjective, so $H_{dk,g}\to\mathbf{P}H^0(\mathbf{P}^3\,,\,\mathscr{O}(d))$ is surjective for some positive integer k, and the image of $H_{dk,g-1}\to\mathbf{P}H^0(\mathbf{P}^3\,,\,\mathscr{O}(d))$ is a proper algebraic subvariety. Let

$$W_{d,k,g} = \{F \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d)) | \exists G \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(k)) \text{ such that the curve } C = \{F=0\} \cap \{G=0\} \text{ is reduced, irreducible and } g(C) \leq g\},$$

$$\widetilde{W}_{d,k,g} = \{\{F, G\} \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d)) \times \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(k)) | \text{ the curve } C = \{F=0\} \cap \{G=0\} \text{ is reduced, irreducible and } g(C) \leq g\}.$$

Since the natural map $H_{dk,g} - W_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ is not dominant by Noether-Lefschetz Theorem, the image of the map $\sigma_2 \colon W_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ contains a Zariski open set. By our assumption, $\sigma_2 \colon W_{d,k,g-1} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ is not dominant. Since the two natural maps $\sigma_1 \colon \widetilde{W}_{d,k,g} \to W_{d,k,g}$, $\sigma_3 \colon \widetilde{W}_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ satisfy $\sigma_3 = \sigma_2 \circ \sigma_1$, there are two sets $W \subset W_{d,k,g} - W_{d,k,g-1}$ and $\widetilde{W} \subset \widetilde{W}_{d,k,g}$, so that the image of the map $\sigma_2 \colon W \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ contains a Zariski open set of $\mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$, and $\sigma_1 \colon \widetilde{W} \to W$ is dominant. Therefore at some regular point of W, we can find a smooth section of $\sigma_1 \colon \widetilde{W} \to W$, that is, there is a pair $\{F, G\} \in \widetilde{W}$, such that for any deformation F_t of F with $F = F_0$ in W, there is an unique deformation G_t of G with $G = G_0$ so that $\{F_t, G_t\} \in \widetilde{W}$. Moreover, we can assume the family of curves $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular, and C_t has a type $\mu(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t) | (i, j) \in \Gamma)$ singularity.

Since the surface $S = \{F = 0\}$ is smooth, we may choose homogeneous coordinates $\{Z_0, Z_1, Z_2, Z_3\}$ for $\mathbf{P}^3$, so that

$$\frac{\partial F}{\partial Z_i}(P_{0j}(0)) \neq 0, \quad Z_i(P_{0j}(0)) \neq 0, \qquad \forall i, (0, j) \in \Gamma.$$

By Lemma 2.4, for any $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$, there is a unique deformation $G' \in H^0(\mathbf{P}^3, \mathscr{O}(k))$ of G constructed above, such that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on S has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0) | (i, j) \in \Gamma)$ singularity.

Consider the case $F' = Z_i U$ with $U \in H^0(\mathbf{P}^3, \mathscr{O}(d-1))$, and let $G' = G'(Z_i U) \in H^0(\mathbf{P}^3, \mathscr{O}(k))$ be the corresponding deformation of G. Since

(2.1)
$$\frac{\partial F}{\partial Z_3}(Z_i G'(Z_j U) - Z_j G'(Z_i U)) = Z_i\left(\frac{\partial F}{\partial Z_3} G'(Z_j U) - \frac{\partial G}{\partial Z_3} Z_j U\right) - Z_j\left(\frac{\partial F}{\partial Z_3} G'(Z_i U) - \frac{\partial G}{\partial Z_3} Z_i U\right),$$

we find that the curve $\{\partial F/\partial Z_3(Z_iG'(Z_jU)-Z_jG'(Z_iU))=0\}$ on S has a weak type μ-1 singularity. But $(\partial F/\partial Z_3)(P_{0s}(0))\neq 0$ for all s by our assumption, so the curve $\{K_{ij}(U)=0\}=\{Z_iG'(Z_jU)-Z_jG'(Z_iU)=0\}$ on S has a weak type $\mu-1$ singularity.

Since $\{F=0\}\cap\{G=0\}$ is reduced and irreducible, it is well known that the polynomial ideal (F, G) generated by F and G satisfies $(F, G)=\sqrt{(F, G)}$. Let K_{k+1} be the space of homogeneous polynomials of degree $k+1$ generated by $K_{ij}(U)$ with $i, j=0, 1, 2, 3$ and

$$U\in H^0(\mathbf{P}^3, \mathscr{O}(d-1)).$$

Case 1. If $\dim(K_{k+1}/(F, G))\geq 2$, we can choose $0\neq Q\in K_{k+1}/(F, G)$ so that the curve $\{Q=0\}$ on S passes through an extra smooth point of $C=\{F=0\}\cap\{G=0\}$. Lemma 2.5 gives

$$dk(k+1)=I(Q, G)_F\geq \sum_{(i,j)\in\Gamma}\mu_{ij}(\mu_{ij}-1)+1,$$

$$\begin{aligned}g(C)&=\frac{1}{2}dk(d+k-4)+1-\sum_{(i,j)\in\Gamma}\frac{1}{2}\mu_{ij}(\mu_{ij}-1)\\&\geq\frac{1}{2}dk(d+k-4)+1-\frac{1}{2}dk(k+1)+\frac{1}{2},\end{aligned}$$

that is, $g(C)\geq\frac{1}{2}dk(d-5)+2$.

Case 2. If $\dim(K_{k+1}/(F, G))=1$, let Q be a generator of $K_{k+1}/(F, G)$. Then $K_{ij}(U)\equiv A_{ij}(U)Q \bmod (F, G)$, where $A_{ij}(U)$ are complex numbers. We may assume $A_{ij}(U)\neq 0$ for some i, j, U. From the construction of $K_{ij}(U)$, we get

$$Z_hK_{ij}(U)+Z_iK_{jh}(u)+Z_jK_{hi}(U)=0,$$
$$(Z_hA_{ij}(U)+Z_iA_{jh}(U)+Z_jA_{hi}(U))Q\equiv 0 \mod (F, G).$$

Since $\{F=0\}\cap\{G=0\}$ is reduced and irreducible, and Q is nontrivial, we must have

$$Z_hA_{ij}(U)+Z_iA_{jh}(U)+Z_jA_{hi}(U)\equiv 0 \mod (F, G).$$

But $\deg F=d\geq 5$, so $\deg G=k=1$. We may assume that $(i, j)=(0, 1)$, i.e., $A_{01}(U)\neq 0$. Then

$$G|A_{01}(U)Z_2+A_{12}(U)Z_0+A_{20}(U)Z_1,$$
$$G|A_{01}(U)Z_3+A_{13}(U)Z_0+A_{30}(U)Z_1,$$

and this is impossible.

Case 3. If $\dim(K_{k+1}/(F, G)) = 0$, then

$$K_{ij}(U) = B_{ij}(U)F + C_{ij}(U)G.$$

Here $B_{ij}(U)$ and $C_{ij}(U)$ are homogeneous polynomials. From the equation

$$Z_h K_{ij}(U) + Z_i K_{jh}(U) + Z_j K_{hi}(U) = 0,$$

it follows that

$$\begin{aligned}(Z_h B_{ij}(U) + Z_i B_{jh}(U) + Z_j B_{hi}(U))F \\ + (Z_h C_{ij}(U) + Z_i C_{jh}(U) + Z_j C_{hi}(U))G = 0.\end{aligned}$$

Since F and G are relative prime, $\deg C_{ij}(U) = 1$, and $\deg F = d \geq 5$, it is easy to see that

$$\begin{aligned}Z_h C_{ij}(U) + Z_i C_{jh}(U) + Z_j C_{hi}(U) = 0, \\ Z_h B_{ij}(U) + Z_i B_{jh}(U) + Z_j B_{hi}(U) = 0,\end{aligned}$$

so that

$$\begin{aligned}C_{ij}(U) = Z_i C_j(U) - Z_j C_i(U), \\ B_{ij}(U) = Z_i B_j(U) - Z_j B_i(U)\end{aligned}$$

for some homogeneous polynomials $B_i(U)$, $C_i(U)$. Therefore

$$\begin{aligned}Z_i G'(Z_j U) - Z_j G'(Z_i U) &= K_{ij}(U) \\ &= (Z_i B_j(U) - Z_j B_i(U))F \\ &\quad + (Z_i C_j(U) - Z_j C_i(U))G,\end{aligned}$$

$$\begin{aligned}Z_i(G'(Z_j U) - B_j(U)F - C_j(U)G) \\ - Z_j(G'(Z_i U) - B_i(U)F - C_i(U)G) = 0,\end{aligned}$$

$$G'(Z_j U) - B_j(U)F - C_j(U)G = Z_j V$$

for some $V \in H^0(\mathbf{P}^3, \mathcal{O}(k-1))$. The curve $\{(\partial F/\partial Z_3)G'(Z_j U) - (\partial G/\partial Z_3)Z_j U = 0\}$ on S has a weak type $\mu - 1$ singularity, $Z_j(P_{0s}(0)) \neq 0$, so we conclude that for any $U \in H^0(\mathbf{P}^3, \mathcal{O}(d-1))$, there is a corresponding $V \in H^0(\mathbf{P}^3, \mathcal{O}(k-1))$, so that the curve $\{(\partial F/\partial Z_3)V - (\partial G/\partial Z_3)U = 0\}$ on S has a weak type $\mu - 1$ singularity. Note that $V = V(U)$ is unique $\bmod (F, G)$.

Now the above argument can be repeated again. We construct the space K_k. If $\dim(K_k/(F, G)) \geq 2$, then as before we get the estimate $g(C) \geq \frac{1}{2}dk(d-4) + 2 \geq \frac{1}{2}dk(d-5) + 2$, while otherwise we may continue on.

If $k \geq d$ and $\dim(K_j/(F, G)) = 0$ for $j = k+1, k, \cdots, k-d+2$, then the above argument will end with a homogeneous polynomial R_3 of degree $k-d$, such that the curve $\{(\partial F/\partial Z_3)R_3 - \partial G/\partial Z_3 \cdot 1 = 0\}$ on S has a weak type $\mu - 1$ singularity. If we replace Z_3 by Z_i $(i = 0, 1, 2)$ and repeat the same argument, then either we get the right estimate for $g(C)$, or we have homogeneous polynomials R_0, R_1, R_2 of degree $k - d$, such that the curve $\{(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \cdot 1 = 0\}$ $(i = 0, 1, 2)$ on S has a weak type $\mu - 1$ singularity. By our construction $R_0 \equiv R_1 \equiv R_2 \equiv R_3 \bmod (F, G)$ and $\deg R_i = k - d < k$, so $R_0 \equiv R_1 \equiv R_2 \equiv R_3 \bmod (F)$. If $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \equiv 0 \bmod (F, G)$ for all i, then $\deg \partial G/\partial Z_i = k - 1 < k$ implies that $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \equiv 0$ $(\bmod F)$, so that the Euler relation will give us $G \equiv 0 \bmod (F)$. Therefore one of $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \not\equiv 0 \bmod (F, G)$, hence $\sum \mu_{ij}(\mu_{ij} - 1) \leq dk(k-1)$ as before, i.e.,

$$g(C) \geq \frac{dk(d-3)}{2} + 1 \geq \frac{dk(d-5)}{2} + 2.$$

If $k < d$ and $\dim(K_j/(F, G)) = 0$ for $j = k+1, k, \cdots, 2$, the above three steps of the argument will end with the following situation: for any $U \in H^0(\mathbf{P}^3, \mathscr{O}(d-k))$, there is a corresponding constant $V = V(U)$, such that the curve $\{(\partial F/\partial Z_3)V - (\partial G/\partial Z_3)U = 0\}$ on S has a weak type $\mu - 1$ singularity. Now we define K_1, and we only need to consider the case $\dim(K_1/(F, G)) = 0$. Take $U = Z_i U'$, and let $V = V(Z_i U')$ be the corresponding constant. Then

$$Z_i V(Z_j U') - Z_j V(Z_i U') = A_{ij}(U')G$$

in K_1, thanks to the fact $\deg F = d \geq 5$. Now

$$(Z_h A_{ij}(U') + Z_i A_{jh}(U') + Z_j A_{hi}(U'))G = 0,$$

and forces $A_{ij}(U') = 0$ for any U', that is $V = V(U') = 0$. Then the curve $\{(\partial G/\partial Z_3)U' = 0\}$ on S has a weak type $\mu - 1$ singularity for any $U' \in H^0(\mathbf{P}^3, \mathscr{O}(d-k-1))$, i.e., the curve $\{\partial G/\partial Z_3 = 0\}$ on S has a weak type $\mu - 1$ singularity. Since $k < d$ and one of the $\partial G/\partial Z_i$ $(i = 0, 1, 2, 3)$ is nontrivial, we get $\sum \mu_{ij}(\mu_{ij} - 1) \leq dk(k-1)$, and

$$g(C) \geq dk(d-5)/2 + 2.$$

This completes the proof of Theorem 2.1.

3. Hyperplane sections of generic surfaces and the proof of Theorem 1

Before we go into the proof of Theorem 1, let us first have a look at the special case $k = 1$. Namely, if C is a hyperplane section of a generic surface in $\mathbf{P}^3$, what kind of singularities can C have?

Proposition 3. *Every hyperplane section of a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ has at most either* (1) 3 *ordinary double points,* (2) *an ordinary double point and a simple cusp* (*locally defined by $x^2 = y^3$*), *or* (3) *a tacnode* (*locally defined by $x^2 = y^4$*).

Proof. We follow the notations in the proof of Theorem 2.1. Let $\{F, G\} \in \widetilde{W}$, and assume $C = \{F = 0\} \cap \{G = 0\}$ has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij})$ singularity. Since for any deformation $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$ of F, there is a deformation $G' \in H^0(\mathbf{P}^3, \mathscr{O}(1))$ of G, such that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on $S = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij})$ singularity, we have

$$\left(\frac{\partial G}{\partial Z_3}F' - \frac{\partial F}{\partial Z_3}G'\right)(P_{0s}) = 0 \tag{3.1}$$

on S for all the singular points P_{0s} on C. If C has at least one double point, then there will be a nontrivial condition imposed on G'. Because of the fact $\deg G = 1$, we may choose homogeneous coordinates $\{Z_0, Z_1, Z_2, Z_3\}$ such that $\partial G/\partial Z_i \neq 0$ for $i = 0, 1, 2, 3$. Note that $P_{0s} \in \{G = 0\}$, $h^0(\mathbf{P}^2, \mathscr{O}(1)) = h^0(\{G = 0\}, \mathscr{O}(1)) = 3$, and that it is well known that any four distinct points of $\mathbf{P}^3$ impose independent conditions on homogeneous polynomials of degree ≥ 3. Thus (3.1) implies that C can be singular at most at three different points.

We show next that there is no point $P \in C$ such that its multiplicity $e(P, C) \geq 3$, i.e., $\mu_{0s} \leq 2$ for all s. Assuming there is one, then for any deformation F_t of $F = F_0$, there is a deformation G_t of $G = G_0$, such that the family of curves $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular and C_t has a singular point $P(t)$ with multiplicity $e(P(t), C_t) \geq 3$. Because $k = 1$ and the surface $\{G_t = 0\}$ is smooth, solving $G_t(1, z_1, z_2, z_3) = 0$, we get $z_3 = \psi_t(z_1, z_2)$, where ψ_t is linear in z_1, z_2. Let

$$f_t(z_1, z_2) = F_t(1, z_1, z_2, \psi_t(z_1, z_2)),$$
$$P(t) = [1, c_1(t), c_2(t), \psi_t(c_1(t), c_2(t))].$$

Then

$$f_t(z_1, z_2) = \sum_{i+j\geq 3} a_{ij}(t)(z_1 - c_1(t))^i(z_2 - c_2(t))^j,$$

$$\frac{df_t}{dt}(z_1, z_2)\bigg|_{t=0} = -\frac{\partial f_0}{\partial z_1}(z_1, z_2)\frac{dc_1(t)}{dt}\bigg|_{t=0} - \frac{\partial f_0}{\partial z_2}(z_1, z_2)\frac{dc_2(t)}{dt}\bigg|_{t=0} + \sum_{i+j\geq 3}\left\{\frac{da_{ij}(t)}{dt}\bigg|_{t=0}\right\}(z_1 - c_1(0))^i(z_2 - c_2(0))^j.$$

As in the proof of Lemma 2.4,

$$\frac{df_t}{dt}(z_1, z_2)\bigg|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'; \tag{3.2}$$

thus

$$\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'\right)(1, z_1, z_2, \psi_0(z_1, z_2)) + \frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} = O(3)$$

at $P(0)$ on $\{G = 0\}$. Since $h^0(\mathbf{P}^2, \mathscr{O}(1)) = 3$, $h^0(\mathbf{P}^2, \mathscr{O}(d)) \geq 6$ for $d \geq 5$, and the set

$$A_2 = \{1, z_1 - c_1(0), z_2 - c_2(0), (z_1 - c_1(0))^2, (z_1 - c_1(0))(z_2 - c_2(0)), (z_2 - c_2(0))^2\}$$

has six elements, so we can choose F', such that the above equation is not true for any choices of $G' \in H^0(\{G = 0\}, \mathscr{O}(1))$ and the two numbers $dc_1(t)/dt|_{t=0}$, $dc_2(t)/dt|_{t=0}$. Therefore C has only double points.

Now we look at the case where C has a simple cusp. Let C_t be a μ-equisingular deformation of C, and $P(t)$ be the simple cusp of C_t. Using the notation of the last paragraph, we have

$$f_t(z_1, z_2) = (a(t)(z_1 - c_1(t)) + b(t)(z_2 - c_2(t)))^2 + \sum_{i+j\geq 3} a_{ij}(t)(z_1 - c_1(t))^i(z_2 - c_2(t))^j,$$

$$\begin{aligned}\frac{df_t}{dt}(z_1, z_2)|_{t=0} = & -\frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}|_{t=0} - \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}|_{t=0} \\ & + \sum_{i+j\geq 3}\left\{\frac{da_{ij}(t)}{dt}|_{t=0}\right\}(z_1 - c_1(0))^i(z_2 - c_2(0))^j \\ & + 2(a(0)(z_1 - c_1(0)) + b(0)(z_2 - c_2(0))) \\ & \cdot\left(\frac{da(t)}{dt}|_{t=0}(z_1 - c_1(0)) + \frac{db(t)}{dt}|_{t=0}(z_2 - c_2(0))\right),\end{aligned}$$

and also, by (3.2),

$$\begin{aligned}&\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'\right)(1, z_1, z_2, \psi_0(z_1, z_2)) \\ &\quad + \frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\ &\quad = 2(a(0)(z_1 - c_1(0)) + b(0)(z_2 - c_2(0))) \\ &\quad \cdot\left(\frac{da(t)}{dt}\bigg|_{t=0}(z_1 - c_1(0)) + \frac{db(t)}{dt}\bigg|_{t=0}(z_2 - c_2(0))\right) + O(3)\end{aligned}$$

at $P = P(0)$ on $\{G = 0\}$. The set A_2 just defined above contains six elements, and we are free to choose $dc_1(t)/dt|_{t=0}$, $dc_2(t)/dt|_{t=0}$, $da(t)/dt|_{t=0}$, and $db(t)/dt|_{t=0}$, so having a simple cusp imposes at least two conditions on G'. Now if D_1 and D_2 are two distinct points of C, one can find hyperplanes H_i $(i = 1, 2)$ so that $H_i = 0$ at D_i and $H_i \neq 0$ at D_j for $j \neq i$. Writing $F' = H_1^3F_1 + H_2^3F_2$, because $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$ and $d \geq 5$, we can choose F_1, F_2 so that the Taylor expansion of $F'|_{G=0}$ has prescribed coefficients up to the second order at any two distinct points $D_1, D_2 \in C$ simultaneously. However $G' \in H^0(\{G = 0\}, \mathscr{O}(1)) = H^0(\mathbf{P}^2, \mathscr{O}(1))$, and $h^0(\mathbf{P}^2, \mathscr{O}(1)) = 3$, so C could not afford two simple cusps. Similarly, writing $F' = H_1F_1 + H_2F_2 + H_1H_2F_3$, we can choose F_1, F_2, F_3 such that $F'|_{G=0}$ has prescribed values at D_1, D_2 and simultaneously its Taylor expansion has prescribed coefficients up to the second order at a point $D_3 \in C$. By (3.1) and above, we see that C cannot have two ordinary double points D_1, D_2 and a simple cusp D_3. So we conclude that if C has no infinitely near point P_{1j} of the first order such that $e(P_{ij}, C) = \mu_{1j} > 1$, then C has at most three nodes or a node and a simple cusp.

Finally, we consider the case that the proper transform of C after blowing up at P_{00} is singular at P_{10}. Let $\{z_1, z_2, z_3\} = \{Z_1/Z_0, Z_2/Z_0, Z_3/Z_0\}$ be local coordinates, and $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ be a

μ-equisingular deformation of C. Keeping f_t, g_t, ψ_t as before, and denoting $\xi = z_1 - c_1(0)$, $\eta = z_2 - c_2(0)/z_1 - c_1(0)$, $P_{00}(t) = [1, c_1(t), c_2(t), \psi_t(c_1(t), c_2(t))]$, $P_{10}(t) = (0, c_3(t))$, we then have

$$\begin{aligned}
f_t(z_1, z_2) &= \sum_{i+j\geq 2} a_{ij}(t)(z_1 - c_1(t))^i (z_2 - c_2(t))^j, \\
\sum_{i+j\geq 2} & a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j \\
&= (z_1 - c_1(0))^2 \left(\sum_{i+j\geq 2} b_{ij}(t)\xi^i (\eta - c_3(t))^j \right) \\
&= (z_1 - c_1(0))^2 f_t^\sharp(\xi, \eta), \\
\frac{df_t}{dt}(z_1, z_2)\bigg|_{t=0} &= -\frac{\partial f_0}{\partial z_1}(z_1, z_2)\, \frac{dc_1(t)}{dt}\bigg|_{t=0} - \frac{\partial f_0}{\partial z_2}(z_1, z_2)\, \frac{dc_2(t)}{dt}\bigg|_{t=0} \\
&\quad + \frac{d}{dt}\left\{ \sum_{i+j\geq 2} a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j \right\}\bigg|_{t=0} \\
&= -\frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} - \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\
&\quad + \frac{d}{dt}((z_1 - c_1(0))^2 f_t^\sharp(\xi, \eta))|_{t=0}, \\
\frac{d}{dt} f_t^\sharp(\xi, \eta)\bigg|_{t=0} &= -\frac{\partial f_0^\sharp}{\partial \eta}\frac{dc_3(t)}{dt}\bigg|_{t=0} + \sum_{i+j\geq 2} \frac{db_{ij}(t)}{dt}\bigg|_{t=0} \xi^i (\eta - c_3(0))^j,
\end{aligned}$$

and also, by (3.2),

$$\begin{aligned}
&\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1} \left(\frac{\partial F}{\partial Z_3}\right) G' \right)(1, z_1, z_2, \psi_0(z_1, z_2)) \\
&\qquad + \frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\
&\quad = (z_1 - c_1(0))^2 \left(-\frac{\partial f_0^\sharp}{\partial \eta}\frac{dc_3(t)}{dt}\bigg|_{t=0} + O(2) \right).
\end{aligned} \tag{3.3}$$

If we take the Taylor expansion of the left side of (3.3) at $z_1 = c_1(0)$, $z_2 = c_2(0)$, then its coefficients of 1, $z_1 - c_1(0)$, $z_2 - c_2(0)$ must be zero.

As we noted early, this imposes at least one condition on G' due to the free choices of $dc_1(t)/dt|_{t=0}$ and $dc_2(t)/dt|_{t=0}$. Since the set $\{1, \xi, \eta - c_3(0)\}$ has three elements, and we are free to choose the number $dc_3(t)/dt|_{t=0}$, if the proper transform of C in the blow-up of S at P_{00} has a double point P_{10}, then at least two more conditions will be imposed on G'. Altogether at least three conditions are imposed on G'. However, $\dim H^0(\{G = 0\}, \mathscr{O}(1)) = 3$, thus it is not hard to see that P_{10} must be an ordinary double point. If P_{10} is a simple cusp, then at least one more condition will be imposed on G' as we have seen in the last paragraph. If we have a worse singularity than a node or a simple cusp at P_{10}, we can go on one more step up as we will do in the proof of Proposition 4 to see that it will impose extra conditions on G'. Therefore P_{00} is a tacnode of C. q.e.d.

Finally we give the

Proof of Theorem 1. Let C be a curve on a generic surface S of degree $d \geq 5$ in $\mathbf{P}^3$. Then C is a complete intersection of S with another surface of degree k. By Theorem 2.1, the geometric genus $g(C) \geq \frac{1}{2}dk(d-5)+2$. For $d \geq 6$, we have

$$g(C) \geq \frac{dk(d-5)}{2} + 2 > \frac{d(d-3)}{2} - 2$$

when $k \geq 2$. We conclude that the sharp lower bound of $g(C)$ can be achieved only by a hyperplane section. When $k = 1$,

$$\begin{aligned} g(C) &= \pi(C) - \sum \frac{\mu_{ij}(\mu_{ij}-1)}{2} \\ &= \frac{d(d-3)}{2} + 1 - \sum \frac{\mu_{ij}(\mu_{ij}-1)}{2} \\ &\geq \frac{d(d-3)}{2} - 2 \end{aligned}$$

by Proposition 3.

It only remains to consider the case $d = 5$. By Theorem 2.1, $g(C) \geq 2$. Our goal is to show that actually we have $g(C) \geq 3$.

Now we assume there is a type $(5, k)$ curve of geometric genus $g(C) = 2$ on a generic quintic surface S. By Proposition 3, we must have $k > 1$. Again we follow the notation in the proof of Theorem 2.1. Let $\{F, G\} \in \widetilde{W}$, and let $C = \{F = 0\} \cap \{G = 0\}$ have a type $\mu = (\mu_{ij}, P_{ij}, E_{ij})$ singularity, such that for any $F' \in H^0(\mathbf{P}^3, \mathscr{O}(5))$, there is a unique $G' = G'(F') \in H^0(\mathbf{P}^3, \mathscr{O}(k))$, so that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on S has a weak type $\mu - 1$ singularity. Let $F_1', F_2' \in H^0(\mathbf{P}^3, \mathscr{O}(5))$.

Then the curve $\{G'(aF_1' + bF_2') - aG'(F_1') - bG'(F_2') = 0\}$ on S has a weak type $\mu - 1$ singularity. We may assume that $G'(aF_1' + bF_2') - aG'(F_1') - bG'(F_2') \equiv 0 \bmod (F, G)$ for all a, b, F_1', F_2'; otherwise we will get $\sum \mu_{ij}(\mu_{ij} - 1) \leq dkk$ by Lemma 2.5, and $g(C) \geq \frac{1}{2}dk(d-4) \geq 3$. Therefore the map $H^0(\mathbf{P}^3, \mathcal{O}(5)) \to H^0(\mathbf{P}^3, \mathcal{O}(k))/(F, G)$, $F' \to G' = G'(F')$ is linear.

Recall that we use K_{k+1} to denote the linear space of homogeneous polynomials of degree $k+1$ generated by $K_{ij}(U) = Z_iG'(Z_jU) - Z_jG'(Z_iU)$ with $i, j = 0, 1, 2, 3$, and $U \in H^0(\mathbf{P}^3, \mathcal{O}(4))$. From the proof of Theorem 2.1 it is easy to see that $\dim(K_{k+1}/(F, G)) \leq 1$ implies $g(C) \geq 3$. So we need only to consider the case where $\dim(K_{k+1}/(F, G)) \geq 2$. As we noted in (1.1), a section of $K_S \otimes C = \mathcal{O}(d+k-4) = \mathcal{O}(k+1)$ with a weak type $\mu - 1$ singularity induces a section of the canonical bundle of the desingularization of C. But $\deg K_{ij}(U) = k+1$, and the curve $\{K_{ij} = 0\}$ on S has a weak type $\mu - 1$ singularity, so $\dim(K_{k+1}/(F, G)) = 2$ because of $g(C) = 2$.

If we fix some $U \in H^0(\mathbf{P}^3, \mathcal{O}(4))$, so that $K_{ij}(U)$ is nontrivial in $K_{ij}/(F, G)$ for some i, j, then the linear span of the set $\{K_{ij}(U) | i, j = 0, 1, 2, 3\}$ is the whole space $K_{k+1}/(F, G)$, as we noted in case 2 of the proof of Theorem 2.1. Let Q_1, Q_2 be two generators of $K_{k+1}/(F, G)$, and

$$Z_iG'(Z_jU) - Z_jG'(Z_iU) = K_{ij}(U) \equiv a_{ij}Q_1 + b_{ij}Q_2 \mod (F, G).$$

Then the 4×4 matrices $A = (a_{ij})$ and $B = (b_{ij})$ are skewsymmetric and nontrivial. If we take a linear transformation $Z_i' = \sum_j h_{ij}Z_j$ of the homogeneous coordinates $\{Z_i\}$, and use the linearity of $F' \to G' = G'(F')$, then

$$Z_i'G'(Z_j'U) - Z_j'G'(Z_i'U) \equiv (HAH^t)_{ij}Q_1 + (HBH^t)_{ij}Q_2 \mod (F, G)$$

with $H = (h_{ij})$. It is well known that we can choose new homogeneous coordinates, still denoted by $\{Z_0, Z_1, Z_2, Z_3\}$, so that the alternative form B has the following standard form:

Case 1:

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(3.4)\qquad Z_hK_{ij}(U)+Z_iK_{jh}(U)+Z_jK_{hi}(U)=0,$$

we have

$$(a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j)Q_1+(b_{ij}Z_h+b_{jh}Z_i+b_{hi}Z_j)Q_2 \equiv 0 \mod (F, G).$$

Setting $\{i, j, h\}=\{1, 2, 3\}$ in (3.4), we get

$$(a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j)Q_1 \equiv 0 \mod (F, G),$$
$$a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j \equiv 0 \mod (F, G).$$

Because $k>1$, $a_{ij}=0$ for $i, j=1, 2, 3$.

Similarly, $a_{ij}=0$ for $i, j=0, 2, 3$. Setting $\{i, j, k\}=\{0, 1, 2\}$ in (3.4), we obtain

$$a_{01}Z_2Q_1+Z_2Q_2 \equiv 0 \mod (F, G),$$

which contradicts the fact that $\deg G=k>1$.

Case 2.

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Setting $\{i, j, h\}=\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}$ in (3.4), we get

$$M_1Q_1+Z_2Q_2 \equiv 0 \mod (F, G),$$
$$M_2Q_1+Z_3Q_2 \equiv 0 \mod (F, G),$$
$$M_3Q_1+Z_0Q_2 \equiv 0 \mod (F, G),$$
$$M_4Q_1+(Z_3+Z_1)Q_2 \equiv 0 \mod (F, G).$$

A linear combination of the above will lead to

$$(3.5)\qquad L_1Q_1+L_2Q_2 \equiv 0 \mod (F, G),$$

where the line $L_2=aZ_0+bZ_1+cZ_2+dZ_3$ with free choices of a, b, c, d. Now we may choose L_2 so that $L_2\cap C$ does not contain any singular points of C, and the intersection number $I_P(L_2, C)_S=1$ at any point P of $L_2\cap C$. By Bezout's Theorem, $L_2\cap C$ contains $5k$ points with at most 2 points in $\{Q_1=0\}\cap C$, because $\deg K_{\tilde{C}}=2g-2=2$ and Q_1 induces a section of $K_{\tilde{C}}$. From $L_1Q_1=-L_2Q_2$ it follows that at least $5k-2$ points of $L_2\cap C$ are on $L_1=0$, so they are on $L_1\cap L_2\cap S$. Since Q_1

and Q_2 are linear independent, (3.5) implies that $L_1 \neq L_2$. We conclude again by Bezout's Theorem that $5k - 2 \leq 5$, i.e., $k = 1$, a contradiction.

This completes the proof of Theorem 1.

4. Subvarieties of higher dimensional hypersurfaces

By the Noether-Lefschetz Theorem, we know that every curve on a generic surface of degree $d \geq 4$ in $\mathbf{P}^3$ is a complete intersection. In higher dimensions we have a better situation, thanks to the Lefschetz Theorem, which states that if V is a hypersurface in $\mathbf{P}^{n+1}$ with $n \geq 3$, then $\operatorname{Pic} V = \mathbb{Z}$, and it is generated by $\mathscr{O}_V(1)$. Now if $M \subset V$ is a codimension-1 subvariety, then it is a complete intersection of V with another hypersurface.

Almost the whole proof of Theorem 1 can be generalized to prove Theorem 2, except we cannot apply intersection theory in higher dimensions; instead we need the following theorem of Hopf (cf. [1, pp. 108]).

Lemma 4.1 (Hopf). *Given any setup of a linear map $\nu: A \otimes B \to C$, where A, B, C are complex vector spaces and ν is injective on each factor separately, then*

$$\dim \nu(A \otimes B) \geq \dim A + \dim B - 1.$$

The analogy of Theorem 2.1 in higher dimensions is the following.

Theorem 4.2. *If M is a codimension-1 subvariety of a generic hypersurface V of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$, and M is a complete intersection of V with another hypersurface of degree k, then*

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

Again the proof of Theorem 4.2 is based on the following three lemmas.

Lemma 4.3. *Let M be a codimension-1 subvariety of a smooth variety V of dimension n, and assume that M has a type $\mu = (\mu_j, X_j, E_j)$ singularity. If $\Omega \subset V$ is an open neighborhood of some point of M, $\{z_1, \cdots, z_n\}$ are local coordinates on Ω, and M is defined by $g(z_1, \cdots, z_n) = 0$ and has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \{0, \cdots, m\})$ singularity on Ω, then the subvariety $\{\partial g(z_1, \cdots, z_n)/\partial z_i = 0\}$ $(i = 1, \cdots, n)$ has a weak type $\mu_\Omega - 1 = (\mu_j - 1, X_j, E_j | j \in \{0, \cdots, m\})$ singularity on Ω.*

Proof. Since the statement of the conclusion is independent of the choice of the local coordinates, we may assume that X_0 is defined locally by $z_{h+1} = \cdots = z_n = 0$. Let

$$z_1' = z_1, \cdots, z_h' = z_h, z_{h+1}' = \frac{z_{h+1}}{z_n}, \cdots, z_{n-1}' = \frac{z_{n-1}}{z_n}, z_n' = z_n$$

be coordinates on the blow-up of Ω along X_0. Then

$$\begin{aligned}
g(z_1, \cdots, z_n) &= g(z'_1, \cdots, z'_h, z'_{h+1}z'_n, \cdots, z'_{n-1}z'_n, z'_n)\\
&= (z'_n)^{\mu_0} g^\sharp(z'_1, \cdots, z'_n),\\
\frac{\partial g}{\partial z_i} &= (z'_n)^{\mu_0}\frac{\partial g^\sharp}{\partial z'_i}, \qquad i = 1, 2, \cdots, h,\\
\frac{\partial g}{\partial z_i} &= (z'_n)^{\mu_0 - 1}\frac{\partial g^\sharp}{\partial z'_i}, \qquad i = h+1, \cdots, n-1,\\
\frac{\partial g}{\partial z_n} &= \mu_0 (z'_n)^{\mu_0-1} g^\sharp + (z'_n)^{\mu_0}\sum \frac{\partial g^\sharp}{\partial z'_i}\frac{\partial z'_i}{\partial z_n}\\
&= \mu_0 (z'_n)^{\mu_0-1} g^\sharp + (z'_n)^{\mu_0-1}\left(- \sum_{i=h+1}^{n-1} z'_i \frac{\partial g^\sharp}{\partial z'_i} + z'_n \frac{\partial g^\sharp}{\partial z'_n}\right).
\end{aligned}$$

Since $\{g^\sharp = 0\}$ has improved singularities, by induction, $\{\partial g^\sharp/\partial z'_i = 0\}$ $(i = 1, \cdots, n)$ has a weak type $(\mu_j - 1, X_j, E_j | j \in \{1, \cdots, m\})$ singularity on the blow-up of Ω along X_0, so $\{\partial g/\partial z_i = 0\}$ $(i = 1, \cdots, n)$ has a weak type $\mu_\Omega - 1$ singularity on Ω.

Lemma 4.4. *If $M_t = \{g_t(z_1, \cdots, z_n) = 0\}$ is a μ-equisingular family of varieties defined in an open set $\Omega \subset \mathbb{C}^n$, and M_t has a type $\mu(t)_\Omega = (\mu_j, X_j(t), E_j(t) | j \in \{0, \cdots, m\})$ singularity on Ω, then the variety $\{dg_t/dt|_{t=0} = 0\}$ has a weak type $\mu(0)_\Omega - 1 = (\mu_j - 1, X_j(0), E_j(0) | j \in \{0, \cdots, m\})$ singularity on Ω.*

Proof. Since $X_0(t)$ is a smooth manifold, we may assume that $X_0(t)$ is locally defined by

$$z_{h+1} = c_{h+1}(z_1, \cdots, z_h, t), \cdots, \qquad z_n = c_n(z_1, \cdots, z_h, t).$$

Then

$$\begin{aligned}
g_t(z_1, \cdots, z_n) = &\sum_{i_{h+1}+\cdots+i_n \geq \mu_0} A_{i_{h+1}, \cdots, i_n}(z_1, \cdots, z_h, t)\\
&\cdot (z_{h+1} - c_{h+1}(z_1, \cdots, z_h, t))^{i_{h+1}} \cdots (z_n - c_n(z_1, \cdots, z_h, t))^{i_n}.
\end{aligned}$$

By replacing Lemma 2.2 by Lemma 4.3, the proof goes exactly in the same way as that of Lemma 2.3.

Lemma 4.5. *Let $F_t \in H^0(\mathbf{P}^{n+1}, \mathcal{O}(d))$, $G_t \in H^0(\mathbf{P}^{n+1}, \mathcal{O}(k))$, and $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ be a μ-equisingular family of varieties with a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \Gamma)$ singularity. Set $dF_t/dt|_{t=0} = F'$, $dG_t/dt|_{t=0} = G'$, and assume that all the hypersurfaces $F_t = 0$ are* *smooth for t in a neighborhood of* 0. *Then the subvariety* $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ $(i = 0, 1, \cdots, n+1)$ *on* $V = \{F_0 = 0\}$ *has a weak type* $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) | j \in \Gamma)$ *singularity, where* $\{Z_0, Z_1, \cdots, Z_{n+1}\}$ *are homogeneous coordinates.*

Proof. For any point $P \in M_0$, we can find an open set $\Omega \ni P$ of V, and generic homogeneous coordinates $\{Z_i'\}$ with $Z_i' = \sum_{j=0}^{n+1} l_{ij} Z_j$ $(i = 0, 1, \cdots, n+1)$, so that $\partial F_0/\partial Z_i' \neq 0$ on Ω for all i. Assuming M_0 has a type $\mu_\Omega(0) = (\mu_j, X_j(0), E_j(0) | j \in \Gamma_\Omega)$ singularity on Ω, and proceeding as in the proof of Lemma 2.4 except using Lemma 4.4 instead of Lemma 2.3, we conclude that the subvariety $\{(\partial F_0/\partial Z_i')G' - (\partial G_0/\partial Z_i')F' = 0\}$ has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Since $(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F'$ is a linear combination of the $(\partial F_0/\partial Z_j')G' - (\partial G_0/\partial Z_j')F'$ $(j = 0, 1, \cdots, n+1)$, and the property of having a weak type $\mu_\Omega(0) - 1$ singularity is additive by §1, we see that $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Selecting a covering of V with open sets, we deduce that the subvariety $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ on V has a weak type $\mu(0) - 1$ singularity.

Proof of Theorem 4.2. As we noted at the beginning of this section, every codimension-1 subvariety of V is a complete intersection. As in $\mathbf{P}^3$, we can find a pair $\{F, G\} \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d)) \times H^0(\mathbf{P}^{n+1}, \mathscr{O}(k))$, which has the following property: both $\{F = 0\}$ and $\{F = 0\} \cap \{G = 0\}$ are reduced and irreducible, and for any deformation F_t of F with $F = F_0$, there is a unique deformation G_t of G with $G = G_0$, so that the family $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular, and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \Gamma)$ singularity.

Now using Lemma 4.5, we may repeat the argument in the proof of Theorem 2.1. We construct the space K_{k+1}, so that for any $K \in K_{k+1}$, $\deg K = k+1$, and the subvariety $\{K = 0\}$ on $V = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_j - 1, X_j(0), E_j(0))$ singularity. By (1.1), a section of $K_V \otimes M = K_V \otimes M_0 = \mathscr{O}(k + d - n - 2)$ with a weak type $\mu - 1$ singularity gives a section of $K_{\widetilde{M}}$. Since

$$\dim(H^0(\mathbf{P}^{n+1}, \mathscr{O}(d - n - 3))/(F, G) = \binom{d-2}{n+1} - \binom{d-k-2}{n+1},$$

if $\dim K_{k+1} \geq 2$, then by Lemma 4.1, we conclude

$$p_g(M) = h^0(\widetilde{M}, K_{\widetilde{M}}) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

If $\dim K_{k+1} \leq 1$, we may follow the argument in the proof of Theorem 2.1 and get the same estimate on $p_g(M)$. q.e.d.

In the special case $k = 1$, we have

Proposition 4. *Let M be a hyperplane section of a generic hypersurface V of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$. Then M has at most $n+1$ singular points, all of which are double points, and the singularity does not affect the geometric genus of M, i.e.,*

$$p_g(M) = \binom{d}{n+1} - \binom{d-1}{n+1}.$$

We postpone the proof of Proposition 4 until the next section. Now Theorem 2 is an easy consequence of Theorem 4.2 and Proposition 4.

Proof of Theorem 2. Let M be a complete intersection of V with another hypersurface of degree k. Then by Theorem 4.2, we have

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

If $k \geq 2$, then

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-4}{n+1} + 1;$$

if $k = 1$, then by Proposition 4, we obtain

$$p_g(M) = \binom{d}{n+1} - \binom{d-1}{n+1}.$$

So

$$p_g(M) \geq \min\left\{\binom{d-2}{n+1} - \binom{d-4}{n+1} + 1, \binom{d}{n+1} - \binom{d-1}{n+1}\right\}.$$

This completes the proof of Theorem 2.

5. Hyperplane sections of generic hypersurfaces in $\mathbf{P}^{n+1}$

In the last section, we saw that if a codimension-1 subvariety $M = \{F=0\} \cap \{G=0\}$ of a generic hypersurface has a type $\mu = (\mu_j, X_j, E_j)$ singularity, then for any deformation F' of F, there is a deformation G' of G, such that the subvariety $\{(\partial G/\partial Z_{n+1})F' - (\partial F/\partial Z_{n+1})G' = 0\}$ on $\{G = 0\}$ has a weak type $\mu - 1$ singularity. Now we are free to choose $F' \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d))$ arbitrarily, and if $\deg G = 1$, then G' must stay in $H^0(\{G=0\}, \mathscr{O}(1))$ with $\dim H^0(\{G=0\}, \mathscr{O}(1)) = n+1$. Thus M cannot afford very bad singularities. Here is a sketch of the

Proof of Proposition 4. We first take a pair

$$\{F, G\} \in H^0(\mathbf{P}^{n+1}, \mathcal{O}(d)) \times H^0(\mathbf{P}^{n+1}, \mathcal{O}(1))$$

as in the proof of Theorem 4.2, and assume that the codimension-1 subvariety $M = \{F = 0\} \cap \{G = 0\}$ of the generic hypersurface $V = \{F = 0\}$ has a type $\mu = (\mu_j, X_j, E_j | j \in \{0, \cdots, m\})$ singularity. Since the hyperplane $\{G = 0\}$ is smooth, we can find homogeneous coordinates $\{Z_0, \cdots, Z_{n+1}\}$ such that $\partial G/\partial Z_i \neq 0$ for $i \in \{0, \cdots, n+1\}$. By Lemma 4.5, we conclude that for any $F' \in H^0(\mathbf{P}^{n+1}, \mathcal{O}(d))$, there is a $G' \in H^0(\mathbf{P}^{n+1}, \mathcal{O}(1))$ so that the variety $\{(\partial G/\partial Z_{n+1})F' - (\partial F/\partial Z_{n+1})G' = 0\}$ on $\{G = 0\}$ has a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j)$ singularity. If P is a singular point of M, we must have

$$\left(\frac{\partial G}{\partial Z_{n+1}} F' - \frac{\partial F}{\partial Z_{n+1}} G'\right)(P) = 0 \tag{5.1}$$

on $\{G = 0\}$. It is well known that homogeneous polynomials of degree $d \geq n+1$ take independent values on any $n+2$ distinct points in $\mathbf{P}^{n+1}$. But $G' \in H^0(\{G = 0\}, \mathcal{O}(1))$, and $h^0(\mathbf{P}^n, \mathcal{O}(1)) = h^0(\{G = 0\}, \mathcal{O}(1)) = n+1$; thus (5.1) implies that M has at most $n+1$ singular points. The same argument as in the proof of Proposition 3 shows that M has no triple points, that is, $\mu_j = 2$ for every j.

By formula (1.1), in order to conclude that the singularity of M does not affect its geometric genus, it suffices to show that $\dim X_j < n-2$ for each j.

Now assume that $\dim X_j = n-2$ for some j. For simplicity, we may assume that M has one double point $P = X_0$, $\dim X_j < n-2$ for $j < m$, $\dim X_m = n-2$, and all points of X_i $(i = 1, \cdots, m)$ are infinitely near points of P.

Given any deformation F_t of F, there is a deformation $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ of $M = \{F = 0\} \cap \{G = 0\}$, so that the family M_t is μ-equisingular and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \{0, 1, \cdots, m\})$ singularity with $\mu_j = 2$ for all j. Let the point $X_0(t) = [1, c_1(t), \cdots, c_{n+1}(t)]$, $z_{0i} = Z_i/Z_0$ for $i = 1, \cdots, n+1$. Solving the equation $G_t = 0$, we get $z_{0(n+1)} = \psi_t(z_{01}, \cdots, z_{0n})$. Set

$$f_{0,t}(z_{01}, \cdots, z_{0n}) = F_t(1, z_{01}, \cdots, z_{0n}, \psi_t(z_{01}, \cdots, z_{0n})),$$

$$\frac{dF_t}{dt}(Z_0, \cdots, Z_{n+1})|_{t=0} = F'(Z_0, \cdots, Z_{n+1}),$$

$$\frac{dG_t}{dt}(Z_0, \cdots, Z_{n+1})|_{t=0} = G'(Z_0, \cdots, Z_{n+1}).$$

Then

$$\frac{df_{0,t}}{dt}\Big|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_{n+1}}\right)^{-1} \frac{\partial F}{\partial Z_{n+1}} G'. \tag{5.2}$$

Since $X_0(t)$ is a double point of $M_t = \{f_{0,t} = 0\}$, we have
(5.3)

$$f_{0,t} = \sum_{i_1+\cdots+i_n\geq 2} a_{i_1\cdots i_n}(t)(z_{01} - c_1(t))^{i_1} \cdots (z_{0n} - c_n(t))^{i_n},$$

$$\frac{df_{0,t}}{dt}\Bigg|_{t=0} = -\sum_{i=1}^{n} \frac{\partial f_{0,0}}{\partial z_{0i}} \cdot \frac{dc_i(t)}{dt}\Bigg|_{t=0}$$
$$+ \left\{ \sum_{i_1+\cdots+i_n\geq 2} \frac{d}{dt} a_{i_1\cdots i_n}(t)(z_{01} - c_1(0))^{i_1} \cdots (z_{0n} - c_n(0))^{i_n} \right\}\Bigg|_{t=0}.$$

Let

$$f_0^*(z_{01}, \cdots, z_{0n}) = \frac{df_{0,t}}{dt}\Bigg|_{t=0} + \sum_{i=1}^{n} \frac{\partial f_{0,0}}{\partial z_{0i}} \cdot \frac{dc_i(t)}{dt}\Bigg|_{t=0}. \tag{5.4}$$

If we write down the Taylor polynomial of f_0^* at the point $X_0(0)$, then its coefficients of 1, $z_{01} - c_1(0)$, $\cdots$, $z_{0n} - c_n(0)$ must all be 0. Since

$$\begin{aligned} &F'(1, z_{01}, \cdots, z_{0n}, \psi_0(z_{01}, \cdots, z_{0n})) \\ &\quad = \sum_{d\geq i_1+\cdots+i_n\geq 0} b_{i_1\cdots i_n}(z_{01} - c_1(0))^{i_1} \cdots (z_{0n} - c_n(0))^{i_n} \end{aligned} \tag{5.5}$$

with free choices of all its coefficients $b_{i_1\cdots i_n}$, the set $\{dc_i(t)/dt|_{t=0} | i = 1, \cdots, n\}$ contains n elements, and f_0^* depends linearly on F', we see that (5.2) and (5.4) imply that there will be at least one condition imposed on G' if M has one double point.

We may move the point $X_0(t) \in V_{0,t} = \{G_t = 0\}$ to $X_0(0) \in \{G = 0\}$ and blow up simultaneously at $X_0(0)$. Let $V_{1,t} \to V_{0,t}$ be the blow-up, $M_{1,t}$ be the proper transform of M_t in $V_{1,t}$, and

$$z_{11} = z_{01} - c_1(0), \quad z_{12} = \frac{z_{02} - c_2(0)}{z_{01} - c_1(0)}, \cdots, \quad z_{1n} = \frac{z_{0n} - c_n(0)}{z_{01} - c_1(0)}$$

be the new coordinates after the blowing up. Then $M_{1,t}$ is defined by $f_{1,t}(z_{11}, \cdots, z_{1n}) = 0$. Here

$$f_{1,t} = \sum_{i_1+\cdots+i_n\geq 2} a_{i_1\cdots i_n}(t) z_{11}^{i_1+\cdots+i_n-2} z_{12}^{i_2} \cdots z_{1n}^{i_n}.$$

By (5.3) and (5.4),

$$\begin{aligned}\frac{df_{1,t}}{dt}\Big|_{t=0} \\ = (z_{01} - c_1(0))^{-2} f_0^*(z_{01}, \cdots, z_{0n}) \\ = z_{11}^{-2} f_0^*(z_{11} + c_1(0), z_{11}\cdot z_{12} + c_2(0), \cdots, z_{11}\cdot z_{1n} + c_n(0)).\end{aligned} \tag{5.6}$$

If we let

$$F_1' = \sum_{d\geq i_1+\cdots+i_n\geq 2} b_{i_1\cdots i_n} z_{11}^{i_1+\cdots+i_n-2} z_{12}^{i_2}\cdots z_{1n}^{i_n},$$

then by (5.5) we can choose $b_{i_1\cdots i_n}$ freely. Furthermore $df_{1,t}/dt|_{t=0}$ depends linearly on F_1' because of (5.2), (5.4), and (5.6). Since $G' \in H^0(\{G=0\}, \mathscr{O}(1))$ and $h^0(\{G=0\}, \mathscr{O}(1)) = n+1$, the main point of rest of the proof is to see what condition

$$\frac{df_{0,t}}{dt}\Big|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_{n+1}}\right)^{-1} \frac{\partial F}{\partial Z_{n+1}} G'$$

must satisfy if M has a certain type of singularity; then we choose an appropriate F' so that there is no G' which satisfies the condition. We need to continue our discussion in the following cases.

Case a. $n=3$. We claim that the proper transform $M_{1,t}$ of M_t in $V_{1,t}$ cannot have more than one singular point on the exceptional divisor $E_0(t)$. Assume that $M_{1,t}$ has two distinct singular double points $P_1(t)$ and $P_2(t)$ on the exceptional divisor $E_0(t)$, and let $P_1(t) = (0, d_1(t), e_1(t))$ and $P_2(t) = (0, d_2(t), e_2(t))$ in the $\{z_{1i}\}$ coordinates. By generic choice of the homogeneous coordinates $\{Z_0, \cdots, Z_4\}$, we may further assume that $d_1(0)\neq d_2(0)$, $e_1(0)\neq e_2(0)$. Since $M_{1,t}$ is defined by $f_{1,t}=0$, we have

$$f_{1,t}(z_{11}, z_{12}, z_{13}) = \sum_{i_1+i_2+i_3\geq 2} c_{i_1i_2i_3}(t) z_{11}^{i_1}(z_{12}-d_1(t))^{i_2}(z_{13}-e_1(t))^{i_3},$$

$$\begin{aligned} f_1^* &= \frac{df_{1,t}}{dt}\Big|_{t=0} + \frac{\partial f_{1,0}}{\partial z_{12}}\frac{dd_1(t)}{dt}\Big|_{t=0} + \frac{\partial f_{1,0}}{\partial z_{13}}\frac{de_1(t)}{dt}\Big|_{t=0} \\ &= \frac{d}{dt}\left\{\sum_{i_1+i_2+i_3\geq 2} c_{i_1i_2i_3}(t) z_{11}^{i_1}(z_{12}-d_1(0))^{i_2}(z_{13}-e_1(0))^{i_3}\right\}\Bigg|_{t=0}.\end{aligned} \tag{5.7}$$

So the coefficients of 1, z_{11}, $z_{12} - d_1(0)$, $z_{13} - e_1(0)$ in the Taylor expansion of f_1^* at $P_1(0)$ must be 0. We have

$$\begin{aligned} F_1' &= \sum_{d\geq i_1+i_2+i_3\geq 2} b_{i_1i_2i_3} z_{11}^{i_1+i_2+i_3-2} z_{12}^{i_2} z_{13}^{i_3} \\ &= \sum_{2\geq i+j\geq 0} b'_{ij}(z_{12}-d_1(0))^i(z_{13}-e_1(0))^j \\ &\quad + z_{11} \sum_{3\geq i+j\geq 0} b''_{ij}(z_{12}-d_1(0))^i(z_{13}-e_1(0))^j + z_{11}^2 \cdot (\cdots). \end{aligned}$$

Here we are free to choose b'_{ij}, b''_{ij}. By (5.7), f_1^* depends on the two numbers $dd_1(t)/dt|_{t=0}$, $de_1(t)/dt|_{t=0}$. Therefore (5.2), (5.5), and (5.6) imply that if $P_1(0)$ is a double point of $M_{1,0}$, then at least two more conditions will be imposed on G'. Similarly the coefficients of 1, $z_{12} - d_2(0)$, and $z_{13} - e_2(0)$ in the Taylor expansion of

$$\left.\frac{df_{1,t}}{dt}\right|_{t=0} + \left.\frac{\partial f_{1,0}}{\partial z_{12}}\frac{dd_2(t)}{dt}\right|_{t=0} + \left.\frac{\partial f_{1,0}}{\partial z_{13}}\frac{de_2(t)}{dt}\right|_{t=0}$$

at $P_2(0)$ must be 0. Moreover any change of the coefficients of $(z_{12}-d_1(0))^2$, $(z_{13}-e_1(0))^2$, $(z_{12}-d_1(0))(z_{13}-e_1(0))$, or $z_{11}(z_{12}-d_1(0))$ of F_1' does not affect the above situation at $P_1(0)$. Since

$$\begin{aligned} (z_{12}-d_1(0))^2 &= 2(d_2(0)-d_1(0))(z_{12}-d_2(0)) \\ &\quad + (z_{12}-d_z(0))^2 + (d_2(0)-d_1(0))^2, \\ (z_{13}-e_1(0))^2 &= 2(e_2(0)-e_1(0))(z_{12}-e_2(0)) \\ &\quad + (z_{13}-e_2(0))^2 + (e_2(0)-e_1(0))^2, \\ (z_{12}-d_1(0))(z_{13}-e_1(0)) &= (d_2(0)-d_1(0))(e_1(0)-e_1(0)) \\ &\quad + (d_2(0)-d_1(0))(z_{13}-e_2(0)) \\ &\quad + (e_2(0)-e_1(0))(z_{12}-d_2(0)) \\ &\quad + (z_{12}-d_2(0))(z_{13}-e_2(0)), \\ z_{11}(z_{12}-d_1(0)) &= (d_1(0)-d_1(0))z_{11} + z_{11}(z_{12}-d_2(0)), \end{aligned}$$

the conditions $d_2(0) \neq d_1(0)$ and $e_2(0) \neq e_1(0)$ imply that we are free to choose the coefficients of 1, z_{11}, $z_{12} - d_2(0)$, $z_{13} - e_2(0)$ of F_1'; thus we are free to choose the coefficients of 1, z_{11}, $z_{12} - d_2(0)$, $z_{13} - e_2(0)$ of f_1^*. Moreover, if $M_{1,0}$ has a second double point $P_2(0)$, then at least two extra conditions will be imposed on G'. But $1 + 2 + 2 > 4 =$

$h^0(\{G=0\}, \mathscr{O}(1))$, so $M_{1,0}$ has at most one singular point. So far if M has a double point, there will be at least one condition imposed on G'. If $M_{1,0}$ has a double point, then two more conditions will be imposed on G'. Since $d \geq 5$, we are free to choose the coefficients of z_{11}^2, z_{11}^3, $z_{11}(z_{12}-d_1(0))$, $z_{11}(z_{13}-e_1(0))$ of F_1'. It is not hard to see that there will be at least two other conditions imposed on G' if the proper transform of $M_{1,0}$ after blowing up at $P_1(0)$ has a double point. Since $h^0(\{G=0\}, \mathscr{O}(1)) = 4$, this is impossible. In conclusion, $\dim X_j = 0$ for every j in case $n=3$.

Case b. $m=1$, that is, $\dim X_1(t) = n-2$, where $X_1(t)$ is a two-fold submanifold of $M_{1,t}$. Since $M_{1,t}$ is defined by $f_{1,t}(z_{11}, \cdots, z_{1n}) = 0$, by Lemma 4.3, $df_{1,t}/dt|_{t=0} = 0$ on $X_1(0)$. Now we can choose all the coefficients of the monomials $1, z_{12}, \cdots, z_{12}^2, z_{12}z_{13}, \cdots, z_{1n}^2$ of F_1' freely, $\dim X_1(0) = n-2$, $h^0(\mathbf{P}^{n-2}, \mathscr{O}(2)) = \binom{n}{2}$, and $df_{1,t}/dt|_{t=0}$ depends linearly on F_1'. Thus the singularity of $M_{1,t}$ along $X_1(t)$ imposes at least $\binom{n}{2}$ conditions on G'. On the other hand, $h^0(\{G=0\}, \mathscr{O}(1)) = n+1 < \binom{n}{2}$ if $n \geq 4$. This is impossible.

Case c. $1 \leq \dim X_1(t) = s_1 < n-2$. Since $M_{1,t}$ has a type $(\mu_j, X_j(t), E_j(t) | j \in \{1, \cdots, m\})$ singularity with $\mu_j = 2$, and $M_{1,t}$ is defined by $f_{1,t} = 0$, by Lemma 4.3, $df_{1,t}/dt|_{t=0} = 0$ has a weak type $(1, X_j(0), E_j(0) | j \in \{1, \cdots, m\})$ singularity. Let us assume that $X_1(0)$ is locally defined by

$$z_{1i} = h_{1i}(z_{1(n-s_1+1)}, \cdots, z_{1n}), \qquad i = 1, \cdots, n-s_1.$$

Rewriting,

$$\begin{aligned} F_1' &= \sum_{d \geq i_1+\cdots+i_n \geq 2} b_{i_1\cdots i_n} z_{11}^{i_1+\cdots+i_n-2} z_{12}^{i_2} \cdots z_{1n}^{i_n} \\ &= \sum b_{i_1\cdots i_n}((z_{11}-h_{11})+h_{11})^{i_1+\cdots+i_n-2}((z_{12}-h_{12})+h_{12})^{i_2} \\ &\quad \cdots ((z_{1(n-s_1)} - h_{1(n-s_1)}) + h_{1(n-s_1)})^{i_{n-s_1}} z_{1(n-s_1+1)}^{i_{n-s_1+1}} \cdots z_{1n}^{i_n} \\ &= F_{1*}'(z_{11} - h_{11}(\cdots), \cdots, z_{1(n-s_1)} - h_{1(n-s_1)}(\cdots), \\ &\qquad z_{1(n-s_1+1)}, \cdots, z_{1n}) + F_{1\sharp}'(z_{1(n-s_1+1)}, \cdots, z_{1n}). \end{aligned} \tag{5.8}$$

Here F_{1*}' is a polynomial of its variables and $F_{1*}'(0, \cdots, 0, z_{1(n-s_1+1)}, \cdots, z_{1n}) = 0$. Since we are free to choose $b_{i_1\cdots i_n}$, we are free to choose the coefficients of the monomials

$$(z_{11}-h_{11}(\cdots))^{i_1}\cdots(z_{1(n-s_1)}-h_{1(n-s_1)}(\cdots))^{i_{n-s_1}}z_{1(n-s_1+1)}^{i_{n-s_1}+1}\cdots z_{1n}^{i_n}$$

of F'_{1*} provided that $i_1+\cdots+i_n\le 2$ and $i_1+\cdots+i_{n-s_1}\ne 0$, and we are also free to choose the coefficients of the monomials $1, z_{1(n-s_1+1)}, \cdots, z_{1n}$, $z^2_{1(n-s_1+1)}, \cdots, z^2_{1n}$ of $F'_{1\sharp}$. Let

$$\frac{df_{1,t}}{dt}|_{t=0}=f'_{1*}+f'_{1\sharp}$$

as in (5.8). Then $df_{1,t}/dt|_{t=0}=0$ on $X_1(0)$ implies that $f'_{1\sharp}\equiv 0$. Since $f_{1\sharp}$ depends linearly on $F'_{1\sharp}$, at least three conditions are imposed on G'. Altogether, we have imposed at least four conditions on G'; this makes up the difference between $h^0(\{G=0\},\mathscr{O}(1))=n+1$ and $\dim X_m(0)=n-2$.

Now let $M_{2,0}$ be the proper transform of $M_{1,0}$ after blowing up along $X_1(0)$, and

$$z_{21}=z_{11}-h_{11}(z_{1(n-s_1+1)},\cdots,z_{1n}),$$
$$z_{2i}=\frac{z_{1i}-h_{1i}(z_{1(n-s_1+1)},\cdots,z_{1n})}{z_{11}-h_{11}(z_{1(n-s_1+1)},\cdots,z_{1n})},\qquad i=2,\cdots,n-s_1,$$
$$z_{2i}=z_{1i},\qquad i=n-s_1+1,\cdots,n,$$

be the new local coordinates. Denoting

$$(5.9)\quad F'_2=z_{21}^{-1}F'_{1*}(z_{21},z_{21}z_{22},\cdots,z_{21}z_{2(n-s_1)},z_{2(n-s_1+1)},\cdots,z_{2n}),$$

we have free choices of the coefficients of $1, z_{21}, \cdots, z_{2n}$ for F'_2. Set

$$(5.10)\quad \begin{aligned} f'_2&=(z_{11}-h_{11}(z_{1(n-s_1+1)},\cdots,z_{1n}))^{-1}\left.\frac{df_{1,t}}{dt}\right|_{t=0}\\ &=z_{21}^{-1}f'_{1*}(z_{21},z_{21}z_{22},\cdots,z_{21}z_{2(n-s_1)},z_{2(n-s_1+1)},\cdots,z_{2n}).\end{aligned}$$

Since $\{df_{1,t}/dt|_{t=0}=0\}$ has a weak type $(1, X_j(0), E_j(0)|j\in\{1,\cdots,m\})$ singularity, by definition, $\{f'_2=0\}$ has a weak type $(1, X_j(0), E_j(0)|j\in\{2,\cdots,m\})$ singularity. Moreover, f'_2 depends linearly on F'_2.

From now on, we will continue our argument inductively. If $\dim X_2(0)=s_2$, we may assume that $X_2(0)$ is locally defined by

$$z_{2(s_2+1)}=h_{2(s_2+1)}(z_{21},\cdots,z_{2s_2}),\cdots,z_{2n}=h_{2n}(z_{21},\cdots,z_{2s_2}),$$

so that we get

$$F_2' = F_{2*}'(z_{21}, \cdots, z_{2s_2}, z_{2(s_2+1)} - h_{2(s_2+1)}, \cdots, z_{2n} - h_{2n})$$
$$+ F_{2\sharp}'(z_{21}, \cdots, z_{2s_2}),$$
$$f_2' = f_{2*}' + f_{2\sharp}'$$

as in (5.8). We are free to choose the coefficients of $z_{2(s_2+1)} - h_{2(s_2+1)}, \cdots, z_{2n} - h_{2n}$ of F_{2*}'. Since we can also choose the coefficients of $1, z_{21}, \cdots, z_{2s_2}$ for $F_{2\sharp}'$ freely, if $f_2' = 0$ holds on $X_2(0)$ (which is equivalent to $f_{2\sharp}' = 0$), then at least $s_2 + 1 = \dim X_2(0) + 1$ conditions will be imposed on G'.

Now if $m = 2$, we have already imposed $4 + \dim X_2(0) + 1 = n + 3$ conditions on G', then we are done. Otherwise, let M_{30} be the proper transform of M_{20} after blowing up along $X_2(0)$, and

$$z_{3i} = z_{2i}, \qquad i = 1, \cdots, s_2,$$
$$z_{3(s_2+1)} = z_{2(s_2+1)} - h_{2(s_2+1)},$$
$$z_{3i} = \frac{z_{2i} - h_{2i}}{z_{2(s_2+1)} - h_{2(s_2+1)}}, \qquad i = s_2 + 2, \cdots, n,$$

be the local coordinates. Denoting

$$f_3' = z_{3(s_2+1)}^{-1} f_{2*}'(z_{31}, \cdots, z_{3(s_2+1)}, z_{3(s_2+1)} z_{3(s_2+2)}, \cdots, z_{3(s_2+1)} z_{3n}),$$
$$F_3' = z_{3(s_2+1)}^{-1} F_{2*}'(z_{31}, \cdots, z_{3(s_2+1)}, z_{3(s_2+1)} z_{3(s_2+2)}, \cdots, z_{3(s_z+1)} z_{3n})$$

as in (5.9) and (5.10), we are free to choose the coefficients of $1, z_{3(s_2+2)}, \cdots, z_{3n}$ for F_3'. Moreover $\{f_3' = 0\}$ has a weak type $(1, X_j(0), E_j(0) | j \in \{3, \cdots, m\})$ singularity, and f_3' depends linearly on F_3'.

For simplicity, let us assume that $X_3(0)$ is locally defined by

$$z_{3i} = h_{3i}(z_{3(s+1)}, \cdots, z_{3(s+s_3)}), \qquad i \in \{1, \cdots, n\} - \{s+1, \cdots, s+s_3\}.$$

If we write down $f_3' = f_{3*}' + f_{3\sharp}'$, $F_3' = F_{3*}' + F_{3\sharp}'$ as before, then we are free to choose the coefficients of $1, z_{3i} (i \in \{s_2+2, \cdots, n\} \cap \{s+1, \cdots, s+s_3\})$ for $F_{3\sharp}'$, and the coefficients of $z_{3i} - h_{3i}$ $(i \in \{s_2 + 2, \cdots, n\} - \{s + 1, \cdots, s + s_3\})$ for F_{3*}'. If $f_3' = 0$ holds on $X_3(0)$, then at least $\rho = 1 + \#\{\{s_2 + 2, \cdots, n\} \cap \{s + 1, \cdots, s + s_3\}\}$ conditions will be imposed on G'. If we construct F_4' inductively, then we are free to choose $(n - s_2 - 1) - (\rho - 1) = n + 1 - [(s_2 + 1 + \rho]$ coefficients of the zero and the first orders of F_4'.

We may continue this argument. Either we have already imposed more than $n+1$ conditions on G' before we have reached $X_m(0)$, or we have imposed $1+3+\lambda \leq n+1$ conditions on G', and we have a free choice of $n+1-\lambda$ coefficients of the zero and the first orders of F'_m (hence f'_m). Since $\dim X_m(0) = n-2$, if $X_m(0)$ is defined by $z_{m1} = h_{m1}(z_{m3}, \cdots, z_{mn})$, $z_{m2} = h_{m2}(z_{m3}, \cdots, z_{mn})$, then $f'_m = f'_{m*} + f'_{m\sharp} = 0$ on $X_m(0)$ implies that $f'_{m\sharp}(z_{m3}, \cdots, z_{mn}) = 0$. But we are free to choose at least $(n+1-\lambda)-2$ of the coefficients of $1, z_{m3}, \cdots, z_{mn}$ of F'_m. If $f'_m = 0$ holds on $X_m(0)$, then at least $n+1-\lambda-2$ conditions will be imposed on G'; this is impossible since $(1+3+\lambda)+(n+1-\lambda-2) = n+3 > h^0(\{G=0\}, \mathscr{O}(1)) = n+1$.

Case d. $\dim X_1(t) = 0$, that is, $X_1(t)$ is a double point of $M_{1,t}$. We see easily as in case (a) that this imposes two conditions on G'. Therefore if $X_0(0)$ is a double point of M_0 and $X_1(0)$ is a double point of $M_{1,0}$, there will be at least three conditions imposed on G'. Now we can construct F'_2 and f'_2 as above. Using the fact that $f'_2 = 0$ has a weak type $(1, X_j(0), E_j(0) | j \in \{2, \cdots, m\})$ singularity, we may repeat the argument of the second part of case (c). Finally this will impose at least $n+2$ (instead of $n+3$ in case (c)) conditions on G', a contradiction.

This completes the proof of Proposition 4.

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths & J. Harris, *Geometry of algebraic curves, Vol. I*, Springer, Berlin, 1985.

[2] J. Carlson, M. Green, P. Griffiths & J. Harris, *Infinitesimal variation of Hodge structures.* I, Compositio Math. **50** (1983) 109–205 .

[3] H. Clemens, *Curves on generic hypersurface*, Ann. Sci. École Norm. Sup. **19** (1986) 629–636 .

[4] H. Clemens, J. Kollár & S. Mori, *Higher dimensional complex geometry*, Astérisque (1988).

[5] L. Ein, *Subvarieties of generic complete intersection*, Invent. Math. **94** (1988) 163–169 .

[6] ——, *Subvarieties of generic complete intersections.* II, preprint.

[7] G. Ellingsrud & S. A. Stromme, *The number of twisted cubic curves on the general quintic threefold*, preprint.

[8] M. Green, *Koszul cohomology and geometry*, Lectures on Riemann Surfaces, World Sci. Publ., Singapore and River Edge, NJ, 1989.

[9] ——, *Koszul cohomology and the geometry of projective varieties.* I, II, J. Differential Geometry **19** (1984) 125–171; **20** (1984) 279–289 .

[10] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[11] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. (2) **79** (1964) 109–203, 205–326 .

[12] S. Iitaka, *Algebraic geometry*, Springer, Berlin, 1982.

[13] S. Katz, *On the finiteness of rational curves on quintic threefolds,* Compositio Math. **60** (1986) 151–162.

[14] S. Mori & S. Mukai, *The uniruledness of the moduli space of curves of genus* 11, Algebraic Geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., Vol. 1016, Springer, Berlin, 1983, 334–353.

[15] D. Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians,* preprint.

[16] B. Teissier, *Résolution simultanée.* II: *Résolution simultanée et cycles évanescents,* Sém. les Singularités des Surface (Palaiseau, 1976–1977), Lecture Notes in Math., Vol. 777, Springer, Berlin, 1980, 82–146.

[17] ——, *Variétés polaires.* II: *Multiplicités polaires, sections planes et conditions de Whitney,* Algebraic Geometry (La Rábida, 1981), Lecture Notes in Math., Vol. 961, Springer, Berlin, 1982, 314–491.

[18] J. Wahl, *Equisingular deformations of plane algeboid curves,* Trans. Amer. Math. Soc. **193** (1974) 143–170.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE